隨機路面之產生
Generation of Random Road Profiles

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摘要

本文回顧兩種被廣泛採用於產生一維路面之方法，亦即成形濾波器與正弦函數近似法，這些路面可用於四分之一（或二分之一）車懸吊系統控制器模擬上。本文主要貢獻在於，吾人發現於成形濾波器法中，產生路面之一階系統之時間常數與路面等級是獨立無關的。為完整性，文中亦詳細推導正弦函數近似法中各個頻率之振幅。

關鍵詞：隨機過程，成形濾波器，正弦函數近似

Abstract

In this work we review two of the most commonly adopted methods, namely shaping filter and sinusoidal approximation, for generating one-dimensional random road profiles, that are used in the simulation of a quarter car (or half car) vehicle suspension control system. The major contribution of this work is the following: for the shaping filter method, it is found that the time constant of the first order transfer function generating the road profile is independent of the grade of road. While for the sinusoidal approximation method, a detail derivation of the amplitude of each sinusoidal function is re-derived for completeness.

keywords: random process, shaping filter, sinusoidal approximation

I. INTRODUCTION

Road profiles taken along a lateral line show the superelevation and crown of the road design, plus rutting and other distress. Longitudinal profiles show the design grade, roughness, and texture (see Figure 1) [1]. In this paper, we focus on the longitudinal profiles, which classifications are based on the International Organization for Standardization (ISO 8606). The ISO has proposed road roughness classification using the PSD (power spectral density) values as shown in Figure 2, and Tables 1 and 2 [2]. Paved roads are generally considered to be among road classes A to D.

Although the generation of random road profiles is not a new topic in the simulation of vehicle suspension system, see e.g. [3-8], while they all suffered from lacking of detail discussions and derivations of the results given therein. Hence it is the goal of this paper to fill up this gap.

This paper is organized as follows. In section II we briefly introduce the classification of road profiles based on the PSD of road profile. Then the concept of generating random profiles by shaping filter is re-visited. In the mean time, the time constant of the associated transfer function is derived. After that, a sinusoidal approximation of the road

Table 1 Road roughness values classified by ISO

	degree of roughness $\Phi(n_0)$ (10^{-6}m^2/(cycle/m)) where n_0=0.1cycle/m		
road class	lower limit	geometric mean	upper limit
A (very good)	-	16	32
B (good)	32	64	128
C (average)	128	256	512
D (poor)	512	1024	2048
E (very poor)	2048	4096	8192

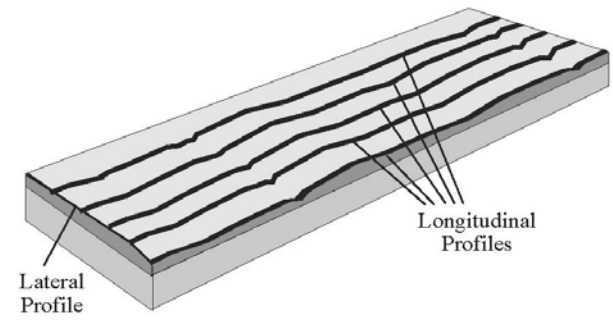

Fig. 1 Road profile

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Table 2 Degree of roughness expressed in terms of ω

	degree of roughness $\Phi(\Omega_0)$ ($10^{-6}m^3$) where Ω_0=1rad/m		
road class	lower limit	geometric mean	upper limit
A (very good)	-	1	2
B (good)	2	4	8
C (average)	8	16	32
D (poor)	32	64	128
E (very poor)	128	256	512

profile is given in section IV. The amplitude of each sinusoidal function is re-derived for completeness here. Finally, two examples are given to demonstrate the resulting grade B profiles generated by the two aforement- ioned methods respectively.

II. CLASSIFICATION OF ROAD PROFILES

The road profile can be represented by a power spectral density (PSD) function. The power spectral densities of roads show a characteristic drop in magnitude with the spatial angular velocity. To determine the PSD, it is necessary to measure the surface profile with respect to a reference plane. Random road profiles can be approximated by a PSD in the form of [2]

$$\Phi(\Omega)=\Phi(\Omega_0)(\frac{\Omega}{\Omega_0})^{-w} \quad \text{or} \quad \Phi(n)=\Phi(n_0)(\frac{n}{n_0})^{-w} \tag{1}$$

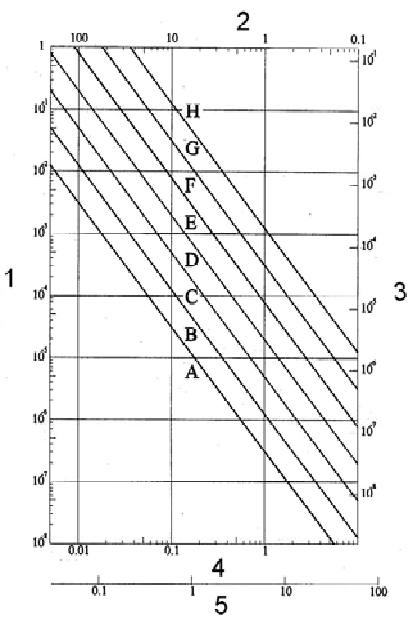

Fig. 2 Road surface classification (ISO 8608). The axes surrounding the frame are defined as 1: displacement psd, Φ(n) [m^3], 2: wavelength, λ[m], 3: displacement psd, Φ(Ω) [m^3], 4: spatial frequency, *n*[cycle/m], 5: angular spatial frequency, Ω [rad/m]

where $\Omega=\frac{2\pi}{L}$ in rad/m denotes the spatial angular velocity, L is the wavelength, $\Phi_0\equiv\Phi(\Omega_0)$ in m^2/(rad/m) describes the values of the psd at the reference spatial angular velocity Ω_0=1 rad/m,
$n=\frac{\Omega}{2\pi}$ is the spatial frequency, $n_0=0.1$ cycle/m,
w is the waviness, for most of the road surface, w=2.

For a rough and quick estimation of the roughness quality, the following guidance is given:
1. new roadway layers, such as, for example, asphalt or concrete layers, can be assumed to have a good or even a very good roughness quality;
2. old roadway layers which are not maintained may be classified as having a medium roughness ;
3. roadway layers consisting of cobblestones or similar material may be classified as medium ("average") or bad ("poor", "very poor").

III. SHAPING FILTER

Before we proceed, some of the fundamental theorems which will be used later are given in the followings [9, 10].

Theorem 3.1 (*Parseval's formula*): If $\overline{X}(\Omega)=F[x(t)]=A(\Omega)e^{j\phi(\Omega)}$ is the Fourier transform of $x(t)$, then

$$\int_{-\infty}^{\infty}|x(t)|^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}A^2(\Omega)d\Omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}|\overline{X}(\Omega)|^2d\Omega$$

Theorem 3.2 (*Wiener-Khintchine Theorem*): Let $R(\tau)$ be an autocorrelation function of $x(t)$, and define the corresponding power spectral density function as:

$$S(\Omega)\equiv\lim_{T\to\infty}E[\frac{1}{2T}|F[X_T(t)]|^2],$$

where $X_T(t)$ is a truncated version of a stationary process $x(t)$,

$$F[X_T(t)]\equiv\int_T^{-T}X(t)e^{-j\Omega\tau}d\tau.$$

Then $S(\Omega)$ is the Fourier transform (if it exists) of $R(\tau)$, that is

$$S(\Omega)\equiv\int_{-\infty}^{\infty}R(\tau)e^{-j\Omega\tau}d\tau \tag{2}$$

and conversely

$$R(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S(\Omega)e^{+j\Omega\tau}d\Omega \tag{3}$$

Remark 3.1: If $X(t)$ is real, then the auto-correlation function $R(\tau)$ is real and even, that is, $R(\tau)=R(-\tau)$, it follows that

$$S(\Omega)=2\int_0^{\infty}R(\tau)\cos(\Omega\tau)d\tau \tag{4}$$

and

$$R(\tau)=\frac{1}{\pi}\int_0^{\infty}S(\Omega)\cos(\Omega\tau)d\Omega \tag{5}$$

To avoid negative angular velocities, usually a one-sided psd is defined with

$$\Phi(\Omega)=\begin{cases}2S(\Omega), & \text{for } \Omega\geq 0\\ 0, & \text{for } \Omega<0\end{cases} \tag{6}$$

Therefore, we obtain

$$R(\tau)=\frac{1}{2\pi}\int_0^\infty \Phi(\Omega)\cos(\Omega\tau)d\Omega \tag{7}$$

1. Road profiles in spatial and temporal domain

It is well known that the amount of road excitation imposed at the vehicle tire depends on two factors:

1-1. the road roughness which is a function of the road roughness coefficient,

1-2. the vehicle velocity V.

Let s be the path variable. By introducing the wavelength

$$\lambda=\frac{2\pi}{\Omega} \tag{8}$$

and assuming that s=0 at t=0, the term Ωs can be written as

$$\Omega s=\frac{2\pi}{\lambda}s=2\pi\frac{V}{\lambda}t=\omega t \tag{9}$$

where ω (rad/sec) is the angular velocity in time domain, we end up with

$$\Omega V=\omega \tag{10}$$

Hence, in the time domain the excitation frequency is given by $f=\frac{\omega}{2\pi}=\frac{V}{\lambda}$. For most of the vehicles the rigid body vibrations are in the range between f =0.5 Hz to f = 15 Hz. This range is covered by waves which satisfy the conditions $0.5\text{Hz}\leq\frac{V}{\lambda}\leq 15\text{Hz}$. For a given wavelength, lets say λ= 4 m, the rigid body vibration of a vehicle are excited if the velocity of the vehicle is varied from $V_{\min}$ = 0.5 Hz · 4 m = 2 m/sec = 7.2 km/h to $V_{\max}$ = 15 Hz · 4 m = 60 m/sec = 216 km/h. Hence, to achieve an excitation in the whole frequency range with moderate vehicle velocities profiles and different varying wavelengths are needed.

When a vehicle is moving along the road with velocity V, the excitation frequency of the road input ω (rad/sec) becomes $\omega=\Omega V$. The mean squared value of road surface roughness, that is the total area of the power spectral density function, does not change with the velocity of a vehicle. Let $\Psi(\omega)$ represent the power spectral density of road input with respect to displacement excitation frequency. Therefore we have the following relation:

$$\Psi(\omega)\,d\omega=\Phi(\Omega)\,d\Omega \tag{11}$$

which in turn yields the relationship between $\Psi(\omega)$ and $\Phi(\Omega)$

$$\Psi(\omega)=\Phi(\Omega)\frac{1}{V} \tag{12}$$

Henceforth, we have

$$\Psi(\omega)=\Phi(\Omega_0)\Omega_0^2\frac{V}{\omega^2} \tag{13}$$

This indicates that the road profile can be obtained from integrating a white noise (i.e. a random walk) in time domain. While to prevent standard deviation from going up with time as the integration period is increased, the road roughness PSD distribution is modified as [3, 4]

$$\Psi(\omega)=\frac{2\alpha V\sigma^2}{\omega^2+\alpha^2V^2} \tag{14}$$

where σ^2 denotes the road roughness variance and V is the vehicle speed, whereas α depends on the type of road surface.

Since the spectral density of the road profile can be factored as

$$\Psi(\omega)=\frac{2\alpha V\sigma^2}{(\alpha V+j\omega)(\alpha V-j\omega)}=H(\omega)\Psi_\omega H^T(-\omega)$$

where $H(\omega)\equiv\frac{1}{\alpha V+j\omega}$ is the frequency response function of the shaping filter, $\Psi_\omega\equiv 2\alpha V\sigma^2$ is the spectral density of a white noise process.

Hence, if the vehicle runs with constant velocity $\frac{ds}{dt}=V$, then the road profile signal, $z_R(t)$, whose PSD is given by equation (14), may be obtained as the output of a linear filter expressed by the differential equation [5, 6]

$$\frac{d}{dt}z_R(t)=-\alpha Vz_R(t)+w(t) \tag{15}$$

where $w(t)$ is a white noise process with the spectral density Ψ_ω. It is easy to see that the profile signal is given by

$$z_R(t)=e^{-\alpha Vt}z_R(0)+\int_0^t e^{-\alpha V(t-\tau)}w(\tau)d\tau$$

and in the steady state the covariance of road irregularities is

$$\lim_{t\to\infty}E[z_R^2(t)]=\frac{\Psi_\omega}{2\alpha V}=\sigma^2 \tag{16}$$

Remark 3.2: Recall that the variance of a continuous process z_R and its corresponding discrete counter part z_{Rk} is related by:

$$E[z_{Rk}z_{Rk}^T]=E[z_Rz_R^T]\cdot\Delta t \tag{17}$$

where Δt is the sampling time.

2. The determination of road profile variance

To prevent PSD from going to infinity in low spatial frequency, a commonly used standard for pavement roughness is the one proposed by ISO [11]. It adopts the following standard formulation to describe pavement roughness PSD:

$$\Phi(\Omega)=\begin{cases}\Phi(\Omega_0)\Omega_1^{-2}, & \text{for } 0\leq\Omega\leq\Omega_1\\ \Phi(\Omega_0)(\frac{\Omega}{\Omega_0})^{-2}, & \text{for } \Omega_1<\Omega\leq\Omega_N\\ 0, & \text{for } \Omega_N<\Omega\end{cases} \tag{18}$$

where the reference values of psd at Ω_0=1 (rad/m), $\Phi(\Omega_0)$, are given by ISO as shown in the Table III. The ISO suggested that $\Omega_1=0.02\pi$ (rad/m), and $\Omega_N=4\pi$ (rad/m) [7, 11].

As a result the variance of the random road profile can be approximated by

$$\sigma^2 = \frac{1}{2\pi}\int_0^\infty \Phi(\Omega)d\Omega = \frac{\Phi(\Omega_0)}{2\pi}\left(\frac{2}{\Omega_1} - \frac{1}{\Omega_N}\right) \approx 5.05\Phi(\Omega_0)$$

But for most of the literature, e.g. [7], it was chosen that $\sigma^2 = 4\Phi(\Omega_0)$.

From equation (14), the one-sided PSD of road profile can be written in spatial frequency as

$$\Phi(\Omega) = \frac{2\alpha\sigma^2}{\Omega^2 + \alpha^2} \tag{19}$$

To determine α, we simply use the relationship

$$\Phi(\Omega_0) = \frac{2\alpha\sigma^2}{\Omega_0^2 + \alpha^2} = \frac{2\alpha \cdot 4\Phi(\Omega_0)}{\Omega_0^2 + \alpha^2} \tag{20}$$

which yields α = 0.127 (rad/m).

Remark 3.3: Note that in this case, α (the reciprocal of time constant [12]) in equation (15) is independent of the road class $\Phi(\Omega_0)$. This agrees with the explanation given in [1, pp.35-36]. While in [3], the author adopted
(1) α=0.15 m^{-1}, σ^2=9 mm^2, V=10-50 m/s that correspond to an asphalt road profile,
(2) α=0.45 m^{-1}, σ^2=300 mm^2, V=5-30 m/s in the case of a paved road.

For the other classes of road, the standard deviation of the corresponding road class can be found in Table 3.

IV. SINUSOIDAL APPROXIMATION

Lemma 4.1: If the vehicle is assumed to travel with a constant speed V over a given road segment with length L, a random profile of a single track can be approximated by a superposition of N (→ ∞) sine waves [8, 10]

$$z_R(s) = \sum_{i=1}^{N} A_i \sin(\Omega_i s - \varphi_i) \tag{21}$$

where the amplitude A_i are defined as follows,

$$A_i \equiv \sqrt{\Phi(\Omega_i)\frac{\Delta\Omega}{\pi}},\ i = 1,..., N \tag{22}$$

in which $\Delta\Omega \equiv \dfrac{\Omega_N - \Omega_1}{N-1}$ (rad/sec), and the phase angles ϕ_i, i = 1,…, N are treated as random variables, following a uniform distribution in the interval $[0,2\pi)$ [13].

Proof: The random process generated by equation (21) can be shown that it has zero mean as follows,

$$\begin{aligned} E[z_R(s)] &= \sum_{i=1}^{N} A_i E[\sin(\Omega_i s - \phi_i)], \\ &= \sum_{i=1}^{N} A_i\{\sin(\Omega_i s)E[\cos\phi_i] - \cos(\Omega_i s)E[\sin\phi_i]\} \\ &= 0 \end{aligned}$$

The variance of the sinusoidal representation is then given by

$$\sigma^2 = \lim_{L\to\infty}\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}\left[\sum_{i=1}^{N} A_i \sin(\Omega_i s - \phi_i)\right]\cdot\left[\sum_{j=1}^{N} A_j \sin(\Omega_j s - \phi_j)\right]ds$$

Table 3 Road roughness standard deviation

Road class	σ (10^{-3}m)	$\Phi(\Omega_0)$(10^{-6}m^3), Ω_0=1	α (rad/m)
A (very good)	2	1	0.127
B (good)	4	4	0.127
C (average)	8	16	0.127
D (poor)	16	64	0.127
E (very poor)	32	256	0.127

For the case i=j :

$$\begin{aligned} J_{ii} &\equiv \int A_i^2 \sin^2(\Omega_i s - \phi_i)ds \\ &= \frac{A_i^2}{2\Omega_i}[\Omega_i s - \phi_i - \frac{1}{2}\sin(2\Omega_i s - 2\phi_i)] \end{aligned}$$

For the case $i \neq j$: use trigonometric relationship,

$$\sin(x)\sin(y) = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$

we have

$$\begin{aligned} J_{ij} &\equiv \int A_i \sin(\Omega_i s - \phi_i)A_j \sin(\Omega_j s - \phi_j)ds \\ &= -\frac{1}{2}\frac{A_i A_j}{\Omega_{i-j}}\sin(\Omega_{i-j}s - \phi_{i-j}) + \frac{1}{2}\frac{A_i A_j}{\Omega_{i+j}}\sin(\Omega_{i+j}s - \phi_{i+j}) \end{aligned}$$

where $\Omega_{i\pm j} \equiv \Omega_i \pm \Omega_j$, $\phi_{i\pm j} \equiv \phi_i \pm \phi_j$.
Henceforth, we have

$$\begin{aligned} \sigma^2 &= \lim_{X\to\infty}\frac{1}{X}\sum_{i=1}^{N}[J_{ii}]_{-\frac{X}{2}}^{\frac{X}{2}} + \lim_{X\to\infty}\frac{1}{X}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ i\neq j}}^{N}[J_{ij}]_{-\frac{X}{2}}^{\frac{X}{2}} \\ &= \sum_{i=1}^{N}\sum_{\substack{j=1 \\ i\neq j}}^{N}\left\{-\frac{A_i A_j}{\Omega_{i-j}}\cos(\phi_{i-j})\lim_{X\to\infty}\frac{\sin(\Omega_{i-j}\frac{X}{2})}{X}\right. \\ &\quad \left. + \frac{A_i A_j}{\Omega_{i+j}}\cos(\phi_{i+j})\lim_{X\to\infty}\frac{\sin(\Omega_{i+j}\frac{X}{2})}{X}\right\} + \sum_{i=1}^{N}\frac{1}{2}A_i^2 = \sum_{i=1}^{N}\frac{1}{2}A_i^2 \end{aligned} \tag{23}$$

Note that the limits of those terms in equation (23) involving sine and cosine are zeros.

Next, we determine the estimate of covariance from PSD. From equation (7) we know that

$$\sigma^2 = R(\xi)|_{\xi=0} = \frac{1}{2\pi}\int_0^\infty \Phi(\Omega)d\Omega \approx \sum_{i=1}^{N}\Phi(\Omega_i)\frac{\Delta\Omega}{2\pi} \tag{24}$$

Comparing equation (23) and (24) indicates that the variance of a sinusoidal approximation to a random road profile can be obtained by letting

$$A_i = \sqrt{\Phi(\Omega_i)\frac{\Delta\Omega}{\pi}},\ i = 1,..., N,$$

where the spatial angular velocities Ω_i are chosen to lie at N equal intervals $\Delta\Omega$.

Remark 4.1: From the PSD defined by equation (1) or (18), and the amplitude A_i given by equation (22), we have

$$A_i \propto \frac{1}{\Omega_i}, \ \text{for} \ \Omega_i >> 1$$

which is similar to the property of Fourier series coefficient of a piecewise continuous function that itself has discontinuity [14].

Remark 4.2: It is easy to see that the road profile can also be generated in the time domain as

$$z_R(t) = \sum_{n=1}^{N} A_n \sin(n\omega_0 t - \phi_n) \tag{25}$$

where the fundamental temporal frequency

$$\omega_0 \equiv V\Delta\Omega, \ \ \Delta\Omega \equiv \frac{2\pi}{L}$$

and A_n is given by equation (22).

V. NUMERICAL EXAMPLE

Example 5.1: Figure 3 depicts the SIMULINK model of the first order filter give by equation (15). A typical output of a grade B road profile generated from the filter is shown in Figure 4. In this case, α=0.127, V=16.6667 (m/sec). In the above figure, the solid line is generated by the SIMULINK model, and the dashed line is created by equation (1). The mean squared error (MSE) of PSD between equation (1) and (15) is $2.35\times10^{-2}\times\Phi(\Omega_0)$.

Example 5.2: A typical grade B road surface created by sinusoidal approximation is shown in Figure 5. The number of terms used is N=512.In this figure, the solid line is obtained by using equation (21), and the dashed line is generated by equation (1). The mean squared error of PSD between equation (21) and (1) is $3.19\times10^{-4}\times\Phi(\Omega_0)$.

As we can tell from Figures 4 and 5, although the appearances of the two profiles look different (this is due to the fact that they share only the same mean value and standard deviation of the profile), both PSDs are close to the desired ones. But strictly speaking, sinusoidal approximation gives a smoother PSD approximation and smaller mean squared error than those of a first order filter.

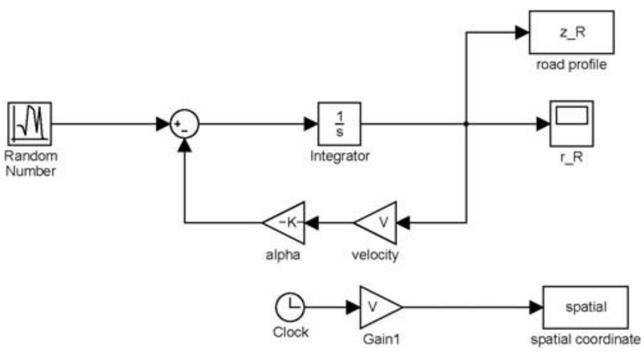

Fig. 3 First order linear system equation (15) Simulink model

VI. CONCLUSIONS

Two of the most commonly adopted methods, namely shaping filter and sinusoidal approximation, for generating random road profiles are reviewed in this work. For the shaping filter we found that the time constant of the associated first-order system transfer function is independent of the road profile grade. However, most of the literature suggests that the time constant is road profile or even vehicle traveling speed dependent. In the sinusoidal approximation, for long enough load profile, we confirmed that the amplitude of each sinusoidal function is proportional to the square root of the related PSD, which is similar to the property of Fourier series coefficient of a function with discontinuity.

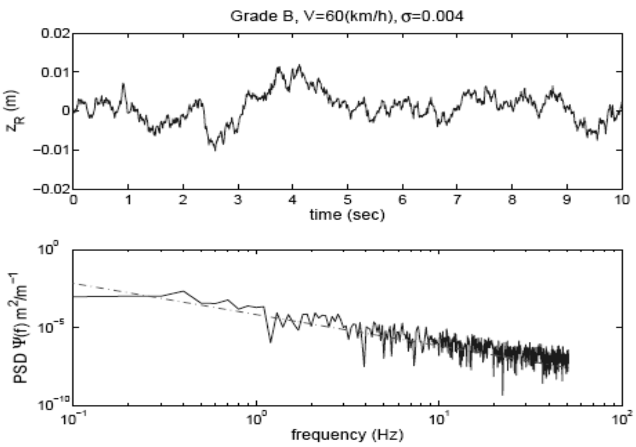

Fig. 4 z_R and the PSD generated from the shaping filter equation (15) (solid line)

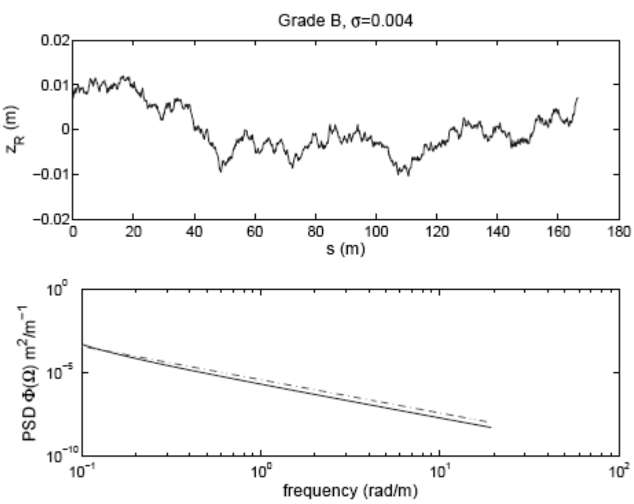

Fig. 5 z_R and the PSD generated from the sinusoidal approximation equation (21) (solid line)

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