拓樸空間具備連續歸屬函數的模糊集
The Fuzzy Sets with Continuous Membership Functions in Topological Spaces

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摘要

本文目的是將模糊集概念從實數空間推廣至拓樸空間。如果明確集是巢狀結構且其集合的包含順序與其索引在實軸上的秩序呈逆向關係，那麼這些明確集可構成模糊集。再者，如果在正規空間裏從開集建造起模糊集，則永遠可以賦予模糊集一連續的歸屬函數。
關鍵詞：模糊集，拓樸空間，連續

Abstract

The objective of this paper is to generalize the concept of fuzzy sets from a real space to a topological space. If the crisp sets are nested and ordered by set inclusion in the reverse way that their subscripts are ordered by the usual ordering in the real line, they can constitute a fuzzy set. Moreover, it is always feasible to construct a fuzzy set which possesses a continuous membership function on a normal space if the crisp set from which a fuzzy set is constructed is a nonempty open set.
Keywords: fuzzy set, topological space, continuous

I. INTRODUCTION

Early mathematical theories were based on certain and infallible reality. Albert Einstein [1] perhaps was one of the first to perceive the dilemma of such an assumption and made a phenomenal statement in 1921: So far as laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality.

In fact, the nature of reality is like a cloud appearing to wander aimlessly but actually following a determined path. In other words, the essence of reality has to possess certain degree of uncertainty. This concept was materialized in 1965 when L. A. Zadeh [2] introduced fuzzy set theory. His underlying idea was to define the degree of membership to a set in terms of a number in the interval [0,1]. In 1967, Goguen [3] generalized this concept with L-fuzzy sets. In spite of the initial lack of support, the fuzzy concept has been eventually evolved as a new kind of mathematics, namely fuzzy topology [4-6].

A great number of fuzzy control applications have emerged ever since E. H. Mamdani and S. Assilian [7] applied the fuzzy logic rules to successfully regulate a model stream engine in 1974. Especially, Japanese industry embraced the idea and developed massive applications in the area of fuzzy logic control such as the subway systems, automobiles, washing machines and robots etc.

Fuzzy numbers [8] are fuzzy sets most useful for engineering applications. Consequently, J. J. Buckley [9] generalized them into fuzzy complex numbers. The continuity of a membership function taken into account in his definition of fuzzy complex numbers motivates us to refine it. Since the continuity of a function depends not only upon the function itself but also on the topologies specified for its domain and range, a substitute for the continuity is the intention of this paper. We will present an explicit way, inspired by the Urysohn's lemma in proving the existence of a continuous function to separate two closed sets, to construct a chain of crisp sets on a normal space. The ingredient of crisp sets to form a fuzzy set whose membership function is continuous is the chain structure bearing on them on a normal space. In a sense, the condition of fuzzy sets with continuous membership functions can be equally replaced by imposing the normality on their universe of discourse and nested structure on their α-cuts.

II. FUZZY SETS

A fuzzy set is a set furnished with a membership function, a degree to which x belongs to the set. In this paper, the level of confidence will be delimited within an unit interval I=[0,1].

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Definition-1: (Fuzzy Set) Let X be the universe of discourse and let I=[0,1] be the level of confidence. If a subset A of X is assigned to a membership function $\mu_A : X \to [0,1]$ such that

$$\mu_A(x) \in (0,1] \text{ for } x \in A \tag{1}$$

and

$$\mu_A(x) = 0 \text{ for } x \notin A \tag{2}$$

The fuzzy set A consists of both its crisp set A and membership function μ_A. Hence, the fuzzy set [2] can be perceived as a set of order pairs of a generic element x and its membership degree $u_A(x)$, i.e.

$$A = \{(x, \mu_A(x)) \mid x \in A\} \tag{3}$$

It is always tempting to refer a new concept on an existed one. Hence, to stratify a fuzzy set into a collection of crisp sets is one way to perceive fuzzy sets under the viewpoint of ordinary sets. Each stratified level of a fuzzy set is called the α-cut. It is defined as follows.

Definition-2: (α-Cut)
The α-cut [1, 4] is defined as a crisp set which includes all elements possessing the degree of the membership greater than or equal to α, i.e. $A_{[\alpha]} = \{x : \alpha \le \mu(x)\}$.

Definition-3: (α-Level Set)
The level set [2] $A_{[\alpha]}$, a fuzzy set introduced by Radecki [10] in 1977, excludes all the points of the fuzzy set A whose membership grades are smaller than α, i.e.

$$A_{[\alpha]} = \{(x, \mu_A(x) \cdot \chi_{A_{[\alpha]}}(x)) \mid x \in A\} \tag{4}$$

Where $X_{A_{[\alpha]}}(x)$ is the characteristic function of $A_{[\alpha]}$.

Definition-4: (α-Layer)
The α-layer [1, 4] of A is a fuzzy set whose membership function is redefined by taking $\alpha \wedge \mu_A(x)$, i.e.

$$\alpha A = \{(x, \alpha \wedge \mu_A(x)) \mid x \in A\} \tag{5}$$

Note that the following symbols are used for brevity.

$$A(x) = \mu_A(x) \tag{6}$$

$$\begin{aligned} \alpha A(x) &= \alpha \wedge \mu_A(x) \\ &= \min(\alpha, \mu_A(x)) \\ &= \begin{cases} \alpha & \text{for } x \in A_{[\alpha]} \\ \mu_A(x) & \text{for } x \notin A_{[\alpha]} \text{ and } x \in A \\ 0 & \text{for } x \notin A \end{cases} \end{aligned} \tag{7}$$

The forementioned definitions are illustrated in Fig.1. The α-layer of an α-level set $A_{[\alpha]}$ is denoted as $\alpha A_{[\alpha]}$, its membership function is computed according to the equation (7) to have

$$\alpha A_{[\alpha]}(x) = \alpha \wedge \mu_{A_{[\alpha]}}(x) = \begin{cases} \alpha & \text{for } x \in A_{[\alpha]} \\ 0 & \text{for } x \notin A_{[\alpha]} \end{cases} \tag{8}$$

The membership function of $\alpha A_{[\alpha]}$ turns out to be equal to the characteristic function of its α-cut multiplied with α, i.e.,

$$\alpha A_{[\alpha]}(x) = \alpha X_{A_{[\alpha]}}(x) \tag{9}$$

In other words, the fuzzy set $\alpha A_{[\alpha]}$ can also be forged by bulging the α-cut $A_{[\alpha]}$ with a height α.

$$\alpha A_{[\alpha]} = \{(x, \alpha X_{A_{[\alpha]}}(x)) \mid x \in A_{[\alpha]}\} \tag{10}$$

A fuzzy set can be naturally decomposed into its α-layers of its α-level sets as shown in Fig.2.

Since the α-layers of a fuzzy set derived from its α-cuts can be unambiguously identified by its α-cuts, the following theorem-1 bridges a fuzzy set to its crisp α-cuts.

Theorem-1: (Representation of a Fuzzy Set)
A fuzzy set [1] can be represented in terms of its α-cuts.

$$A = \vee_{\alpha \in [0,1]} \alpha A_{[\alpha]} \tag{11}$$

Proof:
For any arbitrary $x \in X$, there exists a membership grade, $A(x) = m$ to characterize the possibility of a point, x, belonging to a set, A.

Now, we evaluate the membership grade by its α-cuts.
Since,

$$\alpha A_{[\alpha]}(x) = \begin{cases} \alpha & \text{for } \alpha \in [0, m] \\ 0 & \text{for } \alpha \in (m, 1] \end{cases}.$$

Thus,

$$(\vee \alpha A_{[\alpha]})(x) = \max(\sup_{\alpha \in [0,m]}(\alpha), \sup_{\alpha \in (m,1]}(\alpha)) = m \quad \text{Q.E.D.}$$

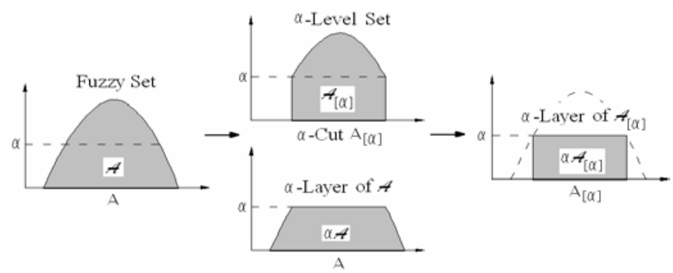

Fig. 1 α-cut, α-Level Set and α-layer

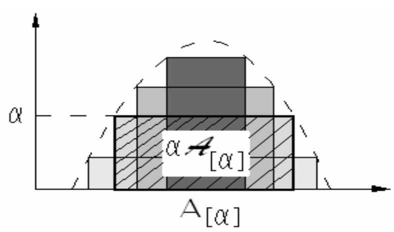

Fig. 2 Decomposition of a Fuzzy Set

Note that this theorem is proclaimed as the first decomposition theorem presented in [1]. The second decomposition theorem [1] states that a fuzzy set can also be represented in terms of its strong α -cuts $A_{(\alpha)} = \{x : \alpha < \mu(x)\}$, that is

$$A = \vee_{\alpha\in[0,1]} \alpha A_{(\alpha)} \tag{12}$$

Although a fuzzy set can be represented in terms of its α -cuts, it doesn't mean that any arbitrary collection of crisp sets is able to constitute a fuzzy set. The condition under which a fuzzy set can be forged from the crisp sets is shown in the following proposition-1.

Proposition-1: Suppose that a collection of nested sets U_α indexed by $\alpha \in I$ possesses a reverse relationship between its set inclusion and indicial order, i.e. $\alpha \le \beta \leftrightarrow U_\alpha \supseteq U_\beta$ and a fuzzy set A is constructed as follows.

$$A = \vee_{\alpha\in[0,1]} \alpha U_\alpha \tag{13}$$

where $U_\alpha = \left\{(x, \alpha X_{U_\alpha}(x)) \mid x \in U_\alpha\right\}$.

Then the strong α -cut $A_{(\alpha)}$ of A is equal to U_α, that is

$$A_{(\alpha)} = U_\alpha \tag{14}$$

Proof:

According to (13), the membership grade is evaluated by $A(x) = \sup_{\beta\in[0,1]} (\beta U_\beta(x))$.

Let ε be a positive small number.

For $\forall \alpha + \varepsilon \in I$ and $\forall x \in U_{\alpha+\varepsilon}$,

$$A(x) = \begin{cases} \alpha+\varepsilon & \text{for } \beta \in [0, \alpha+\varepsilon] \\ \beta & \text{for } \beta \in (\alpha+\varepsilon, 1] \end{cases}.$$

Hence, $A(x) \ge \alpha + \varepsilon \to A(x) > \alpha \to x \in A_{(\alpha)}$.

It implies that $U_{\alpha+\varepsilon} \subset A_{(\alpha)}$ holds for the arbitrary value of ε. By taking $\varepsilon \to 0$, it yields

$$U_\alpha \subset A_{(\alpha)}$$

For $\forall \alpha \in I$ and $\forall x \in A_{(\alpha)}$, define

$$D = \left\{\beta \mid x \in A_{(\alpha)} \cap U_\beta \right\}$$

$\to A(x) = \vee D > \alpha$.

$\to \exists \beta \in D$ s.t. $\beta > \alpha$

For clarity, a pictorial interpretation of the feasible range D is shown in Fig.3 as follows.

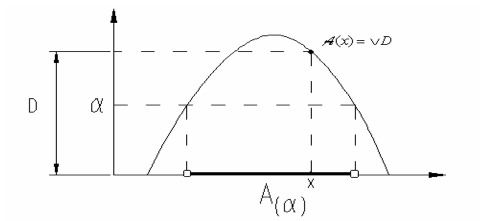

Fig. 3 Feasible Range D of $x \in A_{(\alpha)} \cap U_\beta$

By the assumed condition, $x \in U_\beta \subset U_\alpha$, hence we have

$A_{(\alpha)} \subset U_\alpha$ Q.E.D.

In order to be consistent with the sequel development, the equation (14) uses the strong α -cut instead of α -cut. Since no topology is invoked in the proof, the conclusion of proposition-1 is also valid for α -cut $A_{[\alpha]}$ of A if the provision of ε is given to be zero at the beginning of the proof.

III. CONSTRUCTION OF FUZZY SETS

All related theorems used to prove our assertion that the fuzzy set constructed from a nonempty open set possesses a continuous membership function on the normal space are outlined in this section. The underlying idea mimics the proof of Urysohn's lemma [11] except that the connection between the set inclusion and the subscript of the set is order reversing.

Theorem-2: (Urysohn's Lemma) Let X be a normal space; let A and B be disjoint closed subsets of X. Let [a,b] be a closed interval in the real line. Then there exists a continuous map

$$f : X \to [a,b]$$

such that f(x)=a for every x in A, and f(x)=b for every x in B.

Theorem-3: Every finite point set in a Hausdorff space X is closed.

Proposition-2: A topological space X is T_4 if and only if given any closed subset A of X and any open subset U of X with $A \subset U$, there is an open set V such that $A \subseteq V \subseteq \overline{V} \subseteq U$.

Proposition-3: A topological space X is T_1 if and only if every one-point subset of X is closed.

Theorem-4: (Fuzzy Sets with Continuous Memberships) Let the universe of discourse, X, be a normal space, both T_1 and T_4; let A be any nonempty open subset of X. Then there exists a corresponding fuzzy set A whose membership function μ defined on X is continuous and normal. All of membership function's values lie in the closed unit interval I=[0,1].

Proof:

Since A is a nonempty open set, we pick up a point p from A. A singleton set {p} is a close set in T_1 space. By proposition-2 that is the virtue of normality, there exists an open set $U_{1/2}$ such that

$$\{p\} \subseteq U_{1/2} \subseteq \overline{U}_{1/2} \subseteq A$$

$U_{1/2}$ and A are open sets containing the closed sets $\{p\}$ and $\overline{U}_{1/2}$, respectively. Similarly, there exist open sets $U_{3/4}$ and $U_{1/4}$ such that

$$\overline{U}_1 \subseteq U_{3/4} \subseteq \overline{U}_{3/4} \subseteq U_{1/2} \subseteq \overline{U}_{1/2} \subseteq U_{1/4} \subseteq \overline{U}_{1/4} \subseteq U_0$$

where $\overline{U_1}=\{p\}$ and $U_0=A$.
We continue in this manner and obtain a collection of nested open sets of the form U_α such that $\alpha_1<\alpha_2 \Rightarrow$

$$\overline{U}_{\alpha_2} \subseteq U_{\alpha_1} \qquad (15)$$

where
$\alpha_1,\alpha_2 \in \{\alpha \mid \alpha = m/2^n$, for $n=1,2,3,\ldots$ and $m=1,3,5,\ldots,2^n-1\}$.
Note that m needs to be given only for odd number. In order to prove the above statement valid for every positive integer, we need to invoke the induction on n. Since the cases of n=1,2 have been defined, we assume (15) valid for n. Then we have

$$U_{\frac{(m+1)/2}{2^n}} \subseteq U_{\frac{(m-1)/2}{2^n}} \qquad (16)$$

We can find an open set $U_{\frac{m/2}{2^n}}$ such that

$$U_{\frac{(m+1)/2}{2^n}} \subseteq U_{\frac{m/2}{2^n}} \subseteq \overline{U}_{\frac{m/2}{2^n}} \subseteq U_{\frac{(m-1)/2}{2^n}} \qquad (17)$$

(17) can be rewritten as

$$\overline{U}_{\frac{(m+1)}{2^{n+1}}} \subseteq U_{\frac{m}{2^{n+1}}} \subseteq \overline{U}_{\frac{m}{2^{n+1}}} \subseteq U_{\frac{(m-1)}{2^{n+1}}}.$$

Hence, (15) is valid for every positive integer.
By proposition-1, this collection of nested sets constitutes a fuzzy set.
Let

$$\mu(x)=\begin{cases}1 & for \quad x=p\\ \inf\{\gamma : x\notin U_\gamma\} & for \quad x\neq p\end{cases}$$

Obviously, the values of μ lie in [0,1] and $\mu(p)=1$, $\mu(X-A)=0$. It is clear that $\alpha<\mu(x)$ if and only if x is in some U_γ for $\alpha<\gamma$; and hence it follows that $\mu^{-1}((\alpha,1])=\{x:\alpha<\mu(x)\}=\bigcup_{\gamma\in(\alpha,1]} U_\gamma$, which is open.
Likewise, $\alpha>\mu(x)$ if and only if x is outside of $\overline{U}_\gamma$ for some $\gamma<\alpha$; and hence it follows that $\mu^{-1}([0,\alpha))=\{x:\alpha>\mu(x)\}=\bigcup_{\gamma\in[0,\alpha)}(X-\overline{U}_\gamma)$, which is also open. Thus $\mu(x)$ is continuous. Q.E.D.
space is said to be connected if there does not exist a pair of disjoint nonempty open sets whose union is the space. Since the space on which we live is connected, it does no harm, in fact more practical, to introduce the connectedness into the fuzzy theorem when we deal with the engineering problems. Speaking of connectedness, the simply connected property that every closed path in the space can be shrunk to a point is the best character to generalize the fuzzy numbers from the real space to the topological space. Hence, imposing the connectedness on the fuzzy sets will be a topic of interest in the future development.

IV. CONCLUSIONS

A topological space which is rich in open sets is to guarantee that it is also rich in continuous functions. In this direction, Urysohn's lemma asserts the existence of certain real-valued continuous functions on a normal space. It is the crucial tool used in proving a number of important theorems such as Tietze extension theorem, Urysohn imbedding theorem and imbedding theorem for manifolds. Its underlying idea is to construct a collection of nested sets ordered by inclusion in the same way that their subscripts are ordered by the usual ordering in the real line.

Interestingly, this nested structure also exhibits in the relationship between a fuzzy set and its α-cuts. However, its α-cuts are reversely ordered by its membership value. Hence, it is always feasible to construct a fuzzy set which possesses a continuous membership function on a normal space if the crisp set from which a fuzzy set is constructed is a nonempty open set. If the crisp sets are nested and ordered by set inclusion in the reverse way that their subscripts are ordered by the usual ordering in the real line, they can constitute a fuzzy set with a continuous membership function in a normal space.

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